S.-T. Yau College Student Mathematics Contests 2024

Oral Exams in Geometry and Topology

Individual (Solve 3 out of 4 problems)

1. Let $K \subset \mathbb{S}^3$ be a smoothly embedded circle (we call it a **knot**) in the 3-sphere $\mathbb{S}^3$. Compute all the homology groups of the complement $\mathbb{S}^3 \setminus K$ over $\mathbb{Z}$.

2. (1) Consider the height function z on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Show that its critical points are non-degenerate (namely, the Hessian matrix at the critical point is non-degenerate). Moreover, the number of critical points equals to the rank of its total cohomology $H^*(\mathbb{S}^2, \mathbb{R})$.

(2) Let $\pi : \mathbb{S}^3 \to \mathbb{CP}^1$ be the fibration defined by $[z_1 : z_2]$, where $\mathbb{S}^3$ is the unit sphere in $\mathbb{C}^2$. Consider the pull-back function $\pi^* f$ on $\mathbb{S}^3$. What are the critical points of $\pi^* f$? Are the critical points non-degenerate? If not, can you perturb $\pi^* f$ so that the resulting function only has non-degenerate critical points?

3. Let n, k be positive integers and $2 \leq k \leq n-1$. Suppose S^n is the n-dimensional unit sphere in $\mathbb{R}^{n+1}$ given by

$$\{(x_1, \cdots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

Suppose H is the cone in $\mathbb{R}^{n+1}$ given by

$$\{(x_1, \cdots, x_{n+1} \mid \sum_{i=1}^{k} x_i^2 = c \sum_{i=k+1}^{n+1} x_i^2\},$$

for a constant $c > 0$.

(1) Show that $H \cap S^n$ is a smooth submanifold of S^n.

(2) For what value(s) of c (in terms of n and k) is $H \cap S^n$ a minimal submanifold of S^n?

4. Let a be a positive constant less than 1 and

$$f(u) = (1 + a^2)\frac{1 - u^2}{1 + a^2 u^2} \text{ for } u \in (-1, 1).$$

Consider the Riemmanian metric

$$f(u) d\phi \otimes d\phi + \frac{1}{f(u)} du \otimes du$$

defined on $\{(u, \phi) \mid -1 < u < 1, 0 < \phi < 2\pi\}$.

(1) Show that the metric can be extended to a smooth Riemmanian metric on S^2.

(2) For what value(s) of a can this metric be realized as the induced metric on a closed surface in $\mathbb{R}^3$?